Multiperiodic eigensolutions to the Dirac operator and applications to the generalized Helmholtz equation on flat cylinders and on the n-torus

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Abstract

In this paper we study the solutions to the generalized Helmholtz equation with complex parameter on some conformally flat cylinders and on the n-torus. Using the Clifford algebra calculus, the solutions can be expressed as multiperiodic eigensolutions to the Dirac operator associated to a complex parameter $\lambda \in \mathbb{C}$. Physically, these can be interpreted as the solutions to the time-harmonic Maxwell equations on these manifolds. We study their fundamental properties and give an explicit representation theorem of all these solutions and develop some integral representation formulas. In particular we set up Green type formulas for the cylindrical and toroidal Helmholtz operator. As a concrete application we explicitly solve the Dirichlet problem for the cylindrical Helmholtz operator on the half-cylinder. Finally, we introduce hypercomplex integral operators on these manifolds which allow us to represent the solutions to the inhomogeneous Helmholtz equation with given boundary data on cylinders and on the n-torus.

Keywords: cylindrical and toroidal Helmholtz operator, Helmholtz type equations, Clifford analysis, multi-periodic functions, Dirac operators, Dirichlet type problems, integral operators

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1 Introduction

To introduce the topic of this paper, let us consider a domain $\mathcal{D} \subset \mathbb{R}^3$. Suppose that **E** and **H** are the electrical and magnetic components of an electromagnetic field in $\mathcal{D}$.

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We further suppose that the medium in $\mathcal{D}$ is isotropic and that there are no currents and no charges in $\mathcal{D}$. In the monochromatic case, the electromagnetic field is then governed by the following set of Maxwell equations

$$\begin{array}{ll} \text{rot } \mathbf{H} = \sigma \mathbf{E} & \text{rot } \mathbf{E} = i\omega\mu \mathbf{H} \\ \text{div } \mathbf{H} = 0 & \text{div } \mathbf{E} = 0. \end{array}$$

Here $\sigma := \sigma^* - i\omega\varepsilon$ is the complex electrical conductivity; ε is the dielectric constant; μ is the magnetic permeability and σ^* is the medium electrical conductivity which is the reciprocal value of the electrical resistivity $\rho = \frac{1}{\sigma^*}$. In this case, the electrical and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ obey the homogeneous generalized Helmholtz equations

$$\begin{aligned} \Delta \mathbf{E} - \Lambda \mathbf{E} &= 0 \\ \Delta \mathbf{H} - \Lambda \mathbf{H} &= 0 \end{aligned}$$

where Λ is a complex parameter defined by $\Lambda := -i\omega\mu\sigma^* - \omega^2\mu\varepsilon = -i\omega\mu\sigma \in \mathbb{C}$.
The use of the Clifford algebra calculus makes it possible to express both systems of second order partial differential equations in terms of first order elliptic partial differential operators in an elegant way, viz the factorization

$$(\mathbf{D} - \sqrt{\Lambda})(\mathbf{D} + \sqrt{\Lambda})\mathbf{E} = 0$$

$$(\mathbf{D} - \sqrt{\Lambda})(\mathbf{D} + \sqrt{\Lambda})\mathbf{H} = 0$$

where we choose the branch of $\lambda := \sqrt{\Lambda}$ such that $\Re\lambda > 0$. The number λ is physically interpreted as a medium wave number.

Here $\mathbf{D} := \sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i}$ is the Euclidean three dimensional Dirac operator. This one factorizes the Euclidean Laplacian viz $\mathbf{D}^2 = -\Delta$, when the multiplication is understood as Clifford multiplication in the complex Clifford algebra $Cl_{03}(\mathbb{C})$ which is defined by $e_i^2 = -1$ for $i = 1, 2, 3$ and $e_i e_j = -e_j e_i$ for $i \neq j$.

To solve the original electro-magnetic problem one thus has to study vector valued eigensolutions to the linear first order Euclidean Dirac operator in $\mathbb{R}^3$. The crucial advantage of the treatment of the Maxwell equations in this way consists in the fact that one has a very well-developed function theory for the operator $(\mathbf{D} - \lambda)$. See [4, 9, 10, 16, 17, 18, 29, 31, 33] and elsewhere. Due to the numerous analogies to classical complex function theories, one also calls the solutions to $(\mathbf{D} - \lambda)f = 0$ λ-hyperholomorphic, as for instance in [16, 17, 18]. Actually, the function theoretic techniques allow us to represent the solutions in a very elegant and compact form. Based on these representations one can set up explicit solution algorithms [10]. In [17] it is moreover shown that the operator $\mathbf{D} - \lambda$ when considering $\lambda \in \mathbb{H}(\mathbb{C})$ can also be used to treat the time-harmonic relativistic Dirac operator.
In our recent paper [2] we treated the time-harmonic Maxwell equations on the sphere.

In this paper we want to consider the analogous electromagnetic problem in domains $\mathcal{D}$ that are subdomains of higher dimensional conformally flat tori and cylinders.

Suppose that $\Omega_k \subset \mathbb{R}^n$ is a k-dimensional lattice ($1 \leq k \leq n$) in $\mathbb{R}^n$. Then, cf. [19, 30], the quotient spaces $\mathbb{R}^n/\Omega_k$ are conformally flat cylinders in the case where $k \leq n-1$. In the case $k = n$ we are dealing with the conformally flat n-torus. These are denoted by C_k ($k \leq n-1$) resp. by T_n.

In the three-dimensional case we are dealing with usual infinite cylinders ($k = 1$), a sort of poly-cylinder ($k = 2$) and the 3-torus ($k = 3$). More generally, we can also consider different spinor bundles on these manifolds. Since $\mathbb{R}^n$ is the universal covering space of all these manifolds there exists a well-defined projection map $p_k : \mathbb{R}^n \to C_k$ resp. $p_n : \mathbb{R}^n \to T_n$. This map induces the Helmholtz operator $\Delta'_\lambda := p_k(\Delta + \lambda^2)$ for $k = 1, \ldots, n$ on these cylinders and on the torus. It is called the cylindrical (resp. toroidal) Helmholtz operator. The cylindrical and toroidal Helmholtz operator are factorized by the first order operators $\mathbf{D}'_\lambda$ and $\mathbf{D}_{-\lambda}$. The operator $\mathbf{D}'_\lambda$ is defined by $\mathbf{D}'_\lambda = p_k(\mathbf{D} - \lambda)$ for $k = 1 \ldots, n$. In the three-dimensional case, the null-solutions to $\mathbf{D}'_\lambda$ can physically be interpreted as the solutions to the monochromatic Maxwell equations in domains $\mathcal{D}$ that are sub domains lying on this sort of cylinders and the flat 3-torus, respectively. Our aim is to describe all solutions of the cylindrical and toroidal Maxwell and Helmholtz equations in the 3-dimensional, and more generally, in the n-dimensional case explicitly expressed in terms of numerical series. Mathematically, these series can be regarded as higher dimensional multi-periodic generalizations of the elliptic functions in the context of null-solutions to the operators $\mathbf{D} - \lambda$ resp. $\Delta + \lambda^2$. We perform a basic study of this function class and describe its fundamental properties.

We show that all solutions to the cylindrical and toroidal Maxwell and Helmholtz equations can be expressed in terms of finite sums over basic generalized elliptic functions. Then we develop integral representation theorems for these solutions. These in turn allow us to solve boundary value problems related to the Helmholtz operator on cylinders and on the torus. As a simple concrete example we develop an explicit representation formula for the solutions to the Dirichlet problem for the cylindrical Helmholtz operator on the half-cylinder.
Finally, we introduce hypercomplex integral operators on these manifolds which allow us to represent the solutions to the inhomogeneous generalized Helmholtz equation with given boundary data on cylinders and on the n-torus in a similar way as K. Gürlebeck and W. Sprößig presented in [9, 10] for the Euclidean case.

2 Preliminaries

2.1 Clifford algebras

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of the Euclidean vector space $\mathbb{R}^n$ and $Cl_{0n}(\mathbb{R})$ be the associated real Clifford algebra in which

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, \cdots, n,$$

holds, δ_{ij} standing for the Kronecker symbol. Each element $a \in Cl_{0n}(\mathbb{R})$ can be represented in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$, $A \subseteq \{1, \cdots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n, \quad e_\emptyset = e_0 = 1$. The scalar part of a, denoted by $\mathrm{Sc}(a)$, is defined as the a_0 term. The Clifford conjugate of a is defined by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{l_r} \overline{e}_{l_{r-1}} \cdots \overline{e}_{l_1}$ and $\overline{e}_j = -e_j$ for $j = 1, \cdots, n$, $\overline{e}_0 = e_0 = 1$.

By forming the tensor product $Cl_{0n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ we obtain the complexified Clifford algebra, denoted by $Cl_{0n}(\mathbb{C})$. Its elements are represented in the form $\sum_A a_A e_A$ where the elements a_A are complex numbers of the form $a_A = a_{A1} + i a_{A2}$. The complex imaginary unit i commutes with all basis elements e_j, i.e. we have $i e_j = e_j i$ for all $j = 1, \ldots, n$. We denote the complex conjugate of a complex number $\lambda \in \mathbb{C}$ by $\lambda^\sharp$. For each element $a \in Cl_{0n}(\mathbb{C})$ we have $(\overline{a})^\sharp = \overline{(a^\sharp)}$. On $Cl_{0n}(\mathbb{C})$ one considers a standard (pseudo)norm defined by $\|a\| = (\sum_A |a_A|^2)^{1/2}$. Here $|\cdot|$ is the usual absolute value of the complex number a_A.

2.2 Clifford algebra valued eigensolutions to the Dirac operator

In all that follows we suppose that λ is a complex number. By $\mathbf{D} := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$ we denote the Dirac operator on the n-dimensional Euclidean flat space $\mathbb{R}^n$.

Definition 1 *Let $U \subseteq \mathbb{R}^n$ be an open set and $\lambda \in \mathbb{C}$. Then we say that a function $f : U \to Cl_{0n}(\mathbb{C})$ is λ-holomorphic in U if $(\mathbf{D} - \lambda) f(\mathbf{x}) = 0$ for all $\mathbf{x} \in U$.*

Basic aspects related to the function theory associated to the operator $\mathbf{D} - \lambda$ can be found for instance in [9, 29, 31, 33] and elsewhere. In the case $\lambda = 0$ we deal with the set of left monogenic functions which has been studied extensively by many authors. For its basic theory we refer for instance to [4, 8, 9]. In what follows we assume that $\lambda \neq 0$. Following [33] the fundamental solution to the operator $(\mathbf{D} - \lambda)$ in $\mathbb{R}^n$ has the form

$$e_\lambda(\mathbf{x}) = \begin{cases} \frac{\pi i}{A_n \Gamma(n/2)} \left(\frac{\lambda}{2}\right)^{n/2} \|\mathbf{x}\|^{1-n/2} \Big[H^{(1)}_{n/2-1}(\lambda \|\mathbf{x}\|) - \frac{\mathbf{x}}{\|\mathbf{x}\|} H^{(1)}_{n/2}(\lambda \|\mathbf{x}\|) \Big], & \Im(\lambda) > 0 \\ \frac{-\pi i}{A_n \Gamma(n/2)} \left(\frac{\lambda}{2}\right)^{n/2} \|\mathbf{x}\|^{1-n/2} \Big[H^{(2)}_{n/2-1}(\lambda \|\mathbf{x}\|) - \frac{\mathbf{x}}{\|\mathbf{x}\|} H^{(2)}_{n/2}(\lambda \|\mathbf{x}\|) \Big], & \Im(\lambda) < 0 \\ \frac{\pi}{A_n \Gamma(n/2)} \left(\frac{\lambda}{2}\right)^{n/2} \|\mathbf{x}\|^{1-n/2} \Big[Y_{n/2-1}(\lambda \|\mathbf{x}\|) - \frac{\mathbf{x}}{\|\mathbf{x}\|} Y_{n/2}(\lambda \|\mathbf{x}\|) \Big], & \Im(\lambda) = 0 \end{cases}$$

Here, $A_n = 2\pi^{n/2}/\Gamma(n/2)$ denotes the 'surface area' of the unit ball in $\mathbb{R}^n$. The functions $H^{(1)}$ and $H^{(2)}$ stand for the Hankel functions, defined by

$$\begin{aligned} H^{(1)}_\nu(z) &= J_\nu(z) + i Y_\nu(z), \quad \nu \in \frac{1}{2}\mathbb{N}, \ z \in \mathbb{C} \\ H^{(2)}_\nu(z) &= J_\nu(z) - i Y_\nu(z), \end{aligned}$$

where J and Y stand for the usual Bessel functions of the first and second kind, respectively. See [7] for details.

Following [9, 29, 31, 33] and others, functions satisfying in a domain the equation $(\mathbf{D} - \lambda)f = 0$ obey a Cauchy integral formula of the form

$$f(\mathbf{y}) = \int_{\partial V} e_{-\lambda}(\mathbf{x} - \mathbf{y})n(\mathbf{x})f(\mathbf{x})dS(\mathbf{x})$$

where $V \subset U$ is a compact n-dimensional manifold with a strongly Lipschitz boundary. Further, $n(\mathbf{x})$ denotes the outer normal field and dS is the scalar surface measure of the submanifold ∂V.

We also have the following version of Borel-Pompeiu's formula

$$\int\limits_{\partial V} f(\mathbf{x})n(\mathbf{x})g(\mathbf{x})dS(\mathbf{x}) = \int\limits_{V} [f(\mathbf{x})\mathbf{D}_{-\lambda}]g(\mathbf{x})dV(\mathbf{x}) + \int\limits_{V} f(\mathbf{x})[\mathbf{D}_{\lambda}g(\mathbf{x})dV(\mathbf{x})] \qquad (1)$$

where we set $\mathbf{D}_{\lambda} := \mathbf{D} - \lambda$ and $\mathbf{D}_{-\lambda} := \mathbf{D} + \lambda$.
In the special case putting $f(\mathbf{x}) = e_{-\lambda}(\mathbf{x} - \mathbf{y})$, equation (1) simplifies in view of $e_{-\lambda}(\mathbf{x} - \mathbf{y})(\mathbf{D}_{\mathbf{x}} + \lambda) = \delta(\mathbf{y} - \mathbf{x})$ to

$$\theta(\mathbf{y}) = \Big(\int_{\partial V} e_{-\lambda}(\mathbf{x} - \mathbf{y})n(\mathbf{x})\theta(\mathbf{x})dS(\mathbf{x}) - \int_{V} e_{-\lambda}(\mathbf{x} - \mathbf{y})\mathbf{D}_{\lambda}\theta(\mathbf{x})dS(\mathbf{x})\Big). \qquad (2)$$

Following again [33, 31] and others, eigensolutions to the Dirac operator associated to complex eigenvalues have the following type of Taylor series expansion.

Lemma 1 *(Taylor series expansion). Let $0 < R \leq +\infty$. Let f be a $Cl_{0n}(\mathbb{C})$-valued function that satisfies in the n-dimensional open ball $B(0, R)$ the differential equation $(\mathbf{D}_{\mathbf{x}} - \lambda)f(\mathbf{x}) = 0$ for a complex parameter $\lambda \in \mathbb{C}\backslash\{0\}$. Then there exists a sequence of spherical monogenics of total degree $m = 0, 1, 2, \ldots$, say $P_m(\mathbf{x})$, such that in each open ball $B(0, r)$ with $0 < r < R$*

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} \|\mathbf{x}\|^{1-m-n/2}\Big(J_{m+n/2-1}(\lambda\|\mathbf{x}\|) - \frac{\mathbf{x}}{\|\mathbf{x}\|}J_{m+n/2}(\lambda\|\mathbf{x}\|)\Big)P_m(\mathbf{x}). \qquad (3)$$

The spherical monogenics P_m appearing in this representation are homogeneous monogenic polynomials of total degree m. They have the form

$$P_m(\mathbf{x}) = \sum_{m_2+\cdots+m_n=m} V_{m_2,\ldots,m_n}(\mathbf{x})a_{m_2,\ldots,m_n}$$

where $a_{m_2,\ldots,m_n}$ are Clifford numbers and where $V_{m_2,\ldots,m_n}$ stand for the Fueter polynomials. The latter ones have in the vector formalism the representation

$$V_{m_2,\ldots,m_n}(\mathbf{x}) := \frac{1}{|\mathbf{m}|!}\sum(x_{\sigma(1)} + x_1e_1e_{\sigma(1)})\ldots(x_{\sigma(|\mathbf{m}|)} + x_1e_1e_{\sigma(|\mathbf{m}|)})$$

where $|\mathbf{m}| := m_2 + \cdots + m_n$ and $\sigma(i) \in \{2, \ldots, n\}$. The summation is extended over all distinguishable permutations of the expressions $(x_{\sigma(i)} + x_1e_1e_{\sigma(i)})$ without repetitions.

3 Generalized elliptic functions

In this section we construct n-fold periodic solutions in the kernel of $(\mathbf{D} - \lambda)$. We introduce

Definition 2 *Suppose that $\omega_1, \ldots, \omega_n$ are $\mathbb{R}$-linear independent vectors in $\mathbb{R}^n$. The lattice generated by these vectors will be denoted by $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. Let $S \subset \mathbb{R}^n$ be a closed subset satisfying $S + \omega = S$ for all $\omega \in \Omega$. Furthermore, let $\lambda \in \mathbb{C}$ be a fixed parameter. We call a function $f : \mathbb{R}^n \backslash S \to Cl_{0n}(\mathbb{C})$ that satisfies*

1. *$f(\mathbf{x} + \omega) = f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$*
2. *f is λ-holomorphic except in the points of S*

a λ-holomorphic elliptic function.

In the case $\lambda = 0$ we obtain the monogenic elliptic functions. These were introduced in the 3-dimensional case by A.C. Dixon in [5], in the quaternionic case by R. Fueter [6], in the n-dimensional case by J. Ryan [28] and in complexified Clifford algebras by the second author [13]. Their properties are extensively studied in [12].

3.1 The generalized $\wp$-function for Ker $\mathbf{D} - \lambda$

In this subsection we construct the simplest example of a non-trivial λ-holomorphic elliptic function. First we show

Lemma 2 *Suppose that $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$. Then there exists a real positive constant $c > 0$ such that all $\mathbf{x} \in \mathbb{R}^n$ satisfy the inequality:*

$$\|e_\lambda(\mathbf{x})\|_2 \leq c\sqrt{\frac{2}{|\lambda|\pi}} \frac{e^{-|\Im(\lambda)| \|\mathbf{x}\|_2}}{\sqrt{\|\mathbf{x}\|_2}}. \tag{4}$$

To prove this estimate one relies on the asymptotic relation

$$H_m^{(1)}(\lambda\|\mathbf{x}\|_2) \sim \sqrt{\frac{2}{\pi|\lambda|\|\mathbf{x}\|_2}} e^{i\lambda\|\mathbf{x}\|_2}$$

for all positive parameters $m \in \frac{1}{2}\mathbb{N}$. For simplicity we restrict to consider the orthonormal lattice $\Omega := \mathbb{Z}e_1 + \cdots \mathbb{Z}e_n$. This lattice can be written as the union $\Omega = \bigcup_{m=0}^{+\infty} \Omega_m$ where

$$\Omega_m := \{\omega \in \Omega \mid \|\omega\|_{max} = m\}.$$

We further consider the following subsets of this lattice $L_m := \{\omega \in \Omega \mid \|\omega\|_{max} \leq m\}$. Obviously the set L_m contains exactly $(2m+1)^n$ points. Hence, the cardinality of Ω_m is $\sharp\Omega_m = (2m+1)^n - (2m-1)^n$. The Euclidean distance between the set Ω_{m+1} and Ω_m has the value $d_m := dist_2(\Omega_{m+1}, \Omega_m) = 1$. Now we are able to prove

Theorem 1 *Let $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$. Then the series*

$$\wp_{\lambda;0,\cdots,0}(\mathbf{x}) := \sum_{\omega \in \Omega} e_{\lambda}(\mathbf{x}+\omega) \tag{5}$$

is normally convergent and represents a non-vanishing n-fold periodic function in $\mathbb{R}^n$ satisfying in each point of $\mathbb{R}^n \backslash \Omega$ the differential equation $(\mathbf{D}-\lambda)\wp_{\lambda;0,\cdots,0}(\mathbf{x}) = 0$. In each lattice point $\omega \in \Omega$ it has one pole of the order $n-1$.

Proof. To show the normal convergence of the series, let us consider an arbitrary compact subset $\mathcal{K} \subset \mathbb{R}^n$. Then there exists a positive $r \in \mathbb{R}$ such that all $\mathbf{x} \in \mathcal{K}$ satisfy $\|\mathbf{x}\|_{max} \leq \|\mathbf{x}\|_2 < r$. Suppose now that $\mathbf{x}$ is a point of $\mathcal{K}$. To show the normal convergence of the series, we can leave out without loss of generality a finite set of points. We consider without loss of generality only the summation over those lattice points that satisfy $\|\omega\|_{max} \geq [r]+1$. In view of $\|\mathbf{x}+\omega\|_2 \geq \|\omega\|_2 - \|\mathbf{x}\|_2 \geq \|\omega\|_{max} - \|\mathbf{x}\|_2 = m - \|\mathbf{x}\|_2 \geq m - r$ we obtain

$$\begin{aligned}
& \sum_{m=[r]+1} \sum_{\omega \in \Omega_m} \|e_{\lambda}(\mathbf{x}+\omega)\|_2 \\
\leq \quad & c \sum_{m=[r]+1} \sum_{\omega \in \Omega_m} \sqrt{\frac{2}{|\lambda|\pi}} \frac{e^{-|\Im(\lambda)|\|\mathbf{x}+\omega\|_2}}{\sqrt{\|\mathbf{x}+\omega\|_2}} \\
\leq \quad & c \sum_{m=[r]+1} [(2m+1)^n - (2m-1)^n] \sqrt{\frac{2}{|\lambda|\pi}} \frac{e^{|\Im(\lambda)|(r-m)}}{m-r},
\end{aligned}$$

where c is an appropriately chosen positive real constant, because $m - r \geq [r] + 1 - r > 0$. This sum clearly is absolutely uniformly convergent. Hence, the series $\wp_{\lambda;0,\cdots,0}(\mathbf{x}) := \sum_{\omega \in \Omega} e_{\lambda}(\mathbf{x}+\omega)$, which can be written as

$$\wp_{\lambda;0,\cdots,0}(\mathbf{x}) := \sum_{m=0}^{+\infty} \sum_{\omega \in \Omega_m} e_{\lambda}(\mathbf{x}+\omega),$$

converges normally on $\mathbb{R}^n \backslash \Omega$. Since $e_{\lambda}(\mathbf{x})$ belongs to Ker $(\mathbf{D}-\lambda)$ in each $\mathbf{x} \in \mathbb{R}^n \backslash \{0\}$ and has a pole of order $n-1$ at the origin and exponential decrease for $\|\mathbf{x}\| \to +\infty$, the series $\wp_{\lambda;0,\cdots,0}(\mathbf{x})$ satisfies $(\mathbf{D}-\lambda)\wp_{\lambda;0,\cdots,0}(\mathbf{x}) = 0$ in each $\mathbf{x} \in \mathbb{R}^n \backslash \Omega$ and has a pole of order $n-1$ in each lattice point $\omega \in \Omega$. ■

Remark: Since all re-scaled norms derived from the classical norms are all equivalent on $\mathbb{R}^n$, we also have convergence when taking an arbitrary n-dimensional lattice.

In classical complex analysis, the derivative of the classical Weierstraß $\wp$ function vanishes in all points of the set $\frac{\omega}{2} + \Omega$ where $\omega \in \Omega$ such that $\frac{\omega}{2} \notin \Omega$. This is a consequence of the fact that $\wp$ is an even function and hence $\wp'$ is an odd function, see for instance [8]. The generalized $\wp$ function in the context of Ker $\mathbf{D}-\lambda$ however is neither an even nor an odd function. However, we have the relation $e_{\lambda}(-\mathbf{x}) = \overline{e_{\lambda}(\mathbf{x})}$. As a consequence we can derive the following weaker statement:

Lemma 3 *Let $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$. Let $\omega \in \Omega$ be a lattice point such that $\frac{\omega}{2} \notin \Omega$. Then*

$$\mathrm{Sc}(\wp_{\lambda;0,\cdots,0}(\frac{\omega}{2}+\eta)) = 0 \quad \forall \eta \in \Omega$$

This statement can be derived by applying a direct rearrangement argument of the series which is allowed due to the normal convergence. Notice that this statement remains true for all general n-dimensional lattices that are invariant under the conjugation anti-automorphism, i.e. lattices that satisfy $\overline{\Omega} = \Omega$.

Remark: Let us now consider the case $\lambda \neq 0$ with $\Im(\lambda) = 0$. This case exhibits a different asymptotic behavior than the other cases. Since $Y_\nu(\|\mathbf{x}\|_2) \sim \frac{1}{\sqrt{\|\mathbf{x}\|_2}}$, we have the asymptotic relation

$$\|e_\lambda(\mathbf{x})\|_2 \leq c\sqrt{\frac{2}{|\lambda|\pi}}\frac{\|\mathbf{x}\|_2^{1-n/2}}{\sqrt{\|\mathbf{x}\|_2}} = c\sqrt{\frac{2}{\pi|\lambda|}}\frac{1}{\|\mathbf{x}\|_2^{n/2-1/2}}.$$

Unfortunately, the series

$$\sum_{\omega\in\Omega} e_\lambda(\mathbf{x}+\omega)$$

does not converge in this case. We will get a convergent series representing an n-fold periodic elliptic function for Ker $\mathbf{D} - \lambda$ for real λ if we sum partial derivatives of $e_\lambda(\mathbf{x}+\omega)$ of sufficiently high order over this lattice. This will be mentioned in the next subsection.

3.2 Generalized elliptic functions of higher pole order

Suppose that f is an n-fold periodic function with respect to Ω satisfying $(\mathbf{D}-\lambda)f = 0$ in $\mathbb{R}^n\backslash\Omega$. Let $\mathbf{m} := (m_1,\ldots,m_n) \in \mathbb{N}_0$ be a multi-index with length $|\mathbf{m}| := m_1 + \cdots + m_n$. Then the function $f_{\mathbf{m}}(\mathbf{x}) = \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} f(\mathbf{x})$ is also n-fold periodic with respect to Ω and satisfies $(\mathbf{D}-\lambda)f_{\mathbf{m}}(\mathbf{x}) = 0$. In particular, when taking in the cases $\Im(\lambda) \neq 0$ for f the function $\wp_{\lambda,0\ldots,0}$, then the functions $\wp_{\lambda,\mathbf{m}}(\mathbf{x}) := \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}}\wp_{\lambda;0,\cdots,0}(\mathbf{x})$ are n -fold periodic and satisfy $(\mathbf{D}-\lambda)\wp_{\lambda,\mathbf{m}}(\mathbf{x}) = 0$ in each point of $\mathbb{R}^n\backslash\Omega$. In each lattice point they have an isolated pole of order $n-1+|\mathbf{m}|$. In view of the translation invariance of the operator $\mathbf{D}-\lambda$, we can construct n-fold periodic functions that have poles in a given set of points $a_i + \Omega$ of order N_i ($i = 1,\ldots,l$) with $N_i \geq n-1$ by making the construction

$$\sum_{i=1}^{l} \wp_{\lambda;\mathbf{N}_i}(\mathbf{x}-a_i)b_i \tag{6}$$

where $\mathbf{N}_i$ is a multi-index of length N_i and where b_i are arbitrary elements from $Cl_{0n}(\mathbb{C})$. Due to the compactness of the fundamental period cell, one can only construct λ-holomorphic elliptic functions with a finite number of isolated singularities. It is also possible to construct elliptic functions with non-isolated singularities, as we shall mention below explicitly.

Remark: In the case where $\lambda \neq 0$ with $\Im(\lambda) = 0$ we can construct non-trivial λ-holomorphic elliptic functions by taking $p > \frac{n+1}{2}$ partial derivatives of the fundamental solution $e_\lambda(\mathbf{x}+\omega)$ and summing it over the period lattice. Concretely, we know by Eisenstein's lemma, see e.g. [12] that the series $\sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{\|\omega\|^\alpha}$ converges if $\alpha > n$. Since $e_\lambda(\mathbf{x}) \sim \frac{1}{\|\mathbf{x}\|^{n/2-1/2}}$ we can show by applying the standard arguments from [12] that the series

$$\wp_{\lambda,\mathbf{m}}(\mathbf{x}) := \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} \wp_{\lambda;0,\cdots,0}(\mathbf{x})$$

converges for $\lambda \neq 0$ with $\Im(\lambda) = 0$ if $|\mathbf{m}| > \frac{n+1}{2}$. It has then one pole of order $n-1+|\mathbf{m}|$ in each point of the period lattice. By making the same construction (6) we can construct for the case $\Im(\lambda) = 0$ n-fold periodic λ-holomorphic functions that have poles in a given set of points $a_i + \Omega$ of order N_i $(i = 1, \ldots, l)$ with $N_i > |\mathbf{m}| + n - 1$.

In contrast to the monogenic case treated in [11, 12, 28] and elsewhere, in the λ-holomorphic case with $\Im(\lambda) \neq 0$ it is possible to construct n-fold periodic functions with only one pole of minimal pole order $n-1$ per period cell P. This is not possible in the monogenic case according to the generalized second Liouville theorem, cf. [12]. Notice that also in the λ-holomorphic case, we still have the relation

$$\begin{aligned}
\int_{\partial P} n(\mathbf{x}) f(\mathbf{x}) dS(\mathbf{x}) &= \sum_{j=1}^{n} \Big(\int_{\partial B_j} n(\mathbf{x}) f(\mathbf{x}) dS(\mathbf{x}) + \int_{\partial B'_j} n(\mathbf{x}) f(\mathbf{x}) dS(\mathbf{x}) \Big) \\
&= \sum_{j=1}^{n} \Big(\int_{\partial B_j} n(\mathbf{x}) f(\mathbf{x}+\omega_j) dS(\mathbf{x}) + \int_{\partial B'_j} n(\mathbf{x}) f(\mathbf{x}) dS(\mathbf{x}) \Big) \\
&= \sum_{j=1}^{n} \Big(- \int_{\partial B'_j} n(\mathbf{x}) f(\mathbf{x}) dS(\mathbf{x}) + \int_{\partial B'_j} n(\mathbf{x}) f(\mathbf{x}) dS(\mathbf{x}) \Big) \\
&= 0.
\end{aligned}$$

Here B_j and B'_j denote the opposite surfaces of the n-dimensional fundamental cell with opposite orientation.

However, in the cases $\lambda \neq 0$ the integral $\int_{\partial P} n(\mathbf{x}) f(\mathbf{x}) dS(\mathbf{x})$ has not the meaning of the residue, because we have a different Borel-Pompeiu formula of the form (1). Notice that the function $g = 1$ is not in the kernel of $\mathbf{D} \pm \lambda$ when $\lambda \neq 0$.

In this different context, it naturally arises the question whether one can construct non-trivial λ-holomorphic n-fold periodic functions in $\mathbb{R}^n$. Also here we can prove for the case where $\Im(\lambda) \neq 0$ that the answer is negative.

Lemma 4 *Let $\Im(\lambda) \neq 0$. Suppose that f is an n-fold periodic function that is λ-holomorphic in the whole space $\mathbb{R}^n$. Then f vanishes identically.*

Proof. Since f is n-fold periodic it takes all its values in the fundamental period cell which is compact. Since f is continuous it must be bounded on the fundamental cell. As a consequence of the n-fold periodicity, f must be a bounded function on the whole space $\mathbb{R}^n$. Since f is entire λ-holomorphic the Taylor series representation

$$f(\mathbf{x}) = \sum_{q=0}^{+\infty} \|\mathbf{x}\|^{1-q-n/2} \Big(J_{q+n/2-1}(\lambda\|\mathbf{x}\|) - \frac{\mathbf{x}}{\|\mathbf{x}\|} J_{q+n/2}(\lambda\|\mathbf{x}\|) \Big) P_q(\mathbf{x}) \qquad (7)$$

is valid on the whole space $\mathbb{R}^n$. Since the Bessel functions J are exponentially unbounded away from the real axis the expression f can only be bounded if all spherical monogenics P_q vanish identically. Hence $f \equiv 0$. ■

Notice that all constant functions $f \equiv C$ with $C \neq 0$ are not solutions of $(\mathbf{D} - \lambda)f = 0$.

Finally we can establish the main theorems of this section. It completely describes the set of n-fold periodic λ-holomorphic functions with non-essential singularities in the cases where $\Im(\lambda) \neq 0$.

Theorem 2 *Let $\Im(\lambda) \neq 0$. Let $a_1, a_2, \ldots, a_p \in \mathbb{R}^n \backslash \Omega$ be a finite set of points that are incongruent modulo Ω. Suppose that $f : \mathbb{R}^n \backslash \{a_1 + \Omega, \ldots, a_p + \Omega\} \to Cl_{0n}(\mathbb{C})$ is an n-fold periodic λ-holomorphic function which has at most isolated poles at the points a_i of the order K_i. Then there exist Clifford numbers $b_1, \ldots, b_p \in Cl_{0n}(\mathbb{C})$ such that*

$$f(\mathbf{x}) = \sum_{i=1}^{p} \sum_{m=0}^{K_i-(n-1)} \sum_{m=m_1+m_2+\cdots+m_n} \Big[\wp_{\lambda,\mathbf{m}}(\mathbf{x} - a_i) b_i \Big]. \qquad (8)$$

Proof. Since f is supposed to be λ-holomorphic with isolated poles the points a_i of order K_i its singular parts of the local Laurent series expansions are of the form $e_{\lambda,\mathbf{m}}(\mathbf{x} - a_i)b_i$ in each point $a_i + \Omega$, where $e_{\lambda,\mathbf{m}}(\mathbf{y}) := \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{y}^{\mathbf{m}}} e_\lambda(\mathbf{y})$. As a sum of n-fold periodic λ-holomorphic functions, the expression

$$g(\mathbf{x}) = \sum_{i=1}^{p} \sum_{m=0}^{K_i-(n-1)} \sum_{m=m_1+m_2+\cdots+m_n} \Big[\wp_{\lambda,\mathbf{m}}(\mathbf{x} - a_i) b_i \Big]$$

is also n-fold periodic and has also the same principal parts as f. Hence, the function $h := g - f$ is also n-fold periodic and a λ-holomorphic function, but without singular parts, since these are canceled out. The function h is hence an entire λ-holomorphic n-fold periodic function, and must therefore vanish as a consequence of the preceding lemma. ■

We can adapt this theorem to the case of dealing with n-fold periodic functions that have non-isolated singularities in the following way:

Theorem 3 *Suppose that f is a $Cl_{0n}(\mathbb{C})$ valued n-fold periodic function λ-holomorphic in $\mathbb{R}^n$ except in a finite number of components of non-isolated singular sets $S_1, \ldots, S_l$ of the* *orders $N(S_1),\dots,N(S_l)$ (as defined in [12]) which are supposed to be incongruent modulo Ω. Then there exists functions $b_{\mathbf{j}}: S_j \to Cl_{0n}(\mathbb{C})$ of bounded variation such that*

$$f(\mathbf{x}) = \sum_{i=1}^{l} \sum_{\mathbf{j}\in\mathbf{J}^{(i)}} \Big(\int_{S_i} \wp_{\lambda,\mathbf{j}}(\mathbf{x}-c^{(i)}) d[b_{\mathbf{j}}(c^{(i)})] \Big),$$

where the integral has to be understood as Lebesgue-Stieltjes integral as in [12, 25]. Here we denote by $\mathbf{J}^{(i)}$ the set of those multi indices $\mathbf{j}$ for which the functions $b_{\mathbf{j}}^{(i)}(c^i)$ do not vanish identically.

To establish this more general version one can adapt the arguments of the proof to the previous theorem and the arguments using Lebesgue-Stieltjes integrals as applied in the context of monogenic n-fold periodic functions with non-isolated singularity sets in [12], Chapter 2.

3.3 Elliptic functions to the homogeneous generalized Helmholtz equation

In this section, we restrict to the case $\Im(\lambda) \neq 0$. From the λ-holomorphic n-fold periodic basic function $\wp_{\lambda;0,\cdots,0}$ we can easily obtain n-fold periodic solutions to the Helmholtz operator $(\mathbf{D}+\lambda)(\mathbf{D}-\lambda) = -(\Delta+\lambda^2)$. Let C_1, C_2 be arbitrary Clifford numbers from $Cl_{0n}(\mathbb{C})$. Then the functions

$$\mathrm{Sc}(\wp_{\lambda;0,\dots,0}(\mathbf{x})C_1)$$

and

$$\mathrm{Sc}(\wp_{-\lambda;0,\dots,0}(\mathbf{x})C_2)$$

as well as all its partial derivatives are n-fold periodic and satisfy the homogeneous generalized Helmholtz equation $(\Delta+\lambda^2)f = 0$ in the whole space $\mathbb{R}^n\backslash\Omega$.

We want to obtain a direct generalization of the representation theorem of the previous section for n-fold periodic solutions of the homogeneous Helmholtz equation. To this end we introduce the λ-harmonic basic function

$$h_\lambda(\mathbf{x}) = \begin{cases} \frac{\pi i}{A_n\Gamma(n/2)}\left(\frac{\lambda}{2}\right)^{n/2} \|\mathbf{x}\|^{1-n/2} H^{(1)}_{n/2-1}(\lambda\|\mathbf{x}\|), & \Im(\lambda) > 0 \\ \frac{-\pi i}{A_n\Gamma(n/2)}\left(\frac{\lambda}{2}\right)^{n/2} \|\mathbf{x}\|^{1-n/2} H^{(2)}_{n/2-1}(\lambda\|\mathbf{x}\|), & \Im(\lambda) < 0 \\ \frac{\pi}{A_n\Gamma(n/2)}\left(\frac{\lambda}{2}\right)^{n/2} \|\mathbf{x}\|^{1-n/2} Y_{n/2-1}(\lambda\|\mathbf{x}\|), & \Im(\lambda) = 0 \end{cases}$$

which is nothing else than the scalar part of the function e_λ and has pole order $n-2$ at the origin.

For $\Im(\lambda) \neq 0$ we introduce the scalar valued λ-harmonic Weierstraß $\wp$-function by

$$\wp^h_{\lambda;0,\cdots,0}(\mathbf{x}) := \sum_{\omega\in\Omega} h_\lambda(\mathbf{x}+\omega) \tag{9}$$

For $\lambda \neq 0$ with $\Im(\lambda) = 0$ we get an n-fold periodic λ-harmonic function by taking $|\mathbf{m}| > \frac{n+1}{2}$ partial derivatives of h_λ in this summation. As convergent majorant we again can take the Eisenstein series $\sum_{\omega\in\Omega} \|\omega\|^{-|\mathbf{m}|}$ which converges for $|\mathbf{m}| > \frac{n+1}{2}$.

The analogue of the previous theorem in the λ-harmonic setting has then the form

Theorem 4 *Let $\Im(\lambda) \neq 0$. Let $a_1, a_2, \ldots, a_p \in \mathbb{R}^n \backslash \Omega$ be a finite set of points that are incongruent modulo Ω. Suppose that $f : \mathbb{R}^n \backslash \{a_1 + \Omega, \ldots, a_p + \Omega\} \to Cl_{0n}(\mathbb{C})$ is an n-fold periodic function in the kernel of the Helmholtz operator which has at most isolated poles at the points a_i of the order K_i. Then there exist Clifford numbers $b_1, \ldots, b_p \in Cl_{0n}(\mathbb{C})$ such that*

$$f(\mathbf{x}) = \sum_{i=1}^{p} \sum_{m=0}^{K_i-(n-2)} \sum_{m=m_1+m_2+\cdots+m_n} \left[\wp^h_{\lambda,\mathbf{m}}(\mathbf{x} - a_i) b_i\right]. \tag{10}$$

Similarly, one can adapt this formula to the case dealing with non-isolated singularities.

4 Applications to boundary value problems on the n-dimensional conformally flat torus

4.1 Construction of spinor bundles and sections on the flat n-torus

The space $\mathbb{R}^n$ is the universal covering space of the conformally flat n-torus defined by $\mathbb{R}^n/\Omega$ denoted by T_n. Consequently, there exists a well-defined projection map $p_n : \mathbb{R}^n \to T_n$. As in [14], we call an open subset $U \subset \mathbb{R}^n$ n-fold periodic if for each each $\mathbf{x} \in U$ the point $x + \omega \in U$ for every $\omega \in \Omega$. Then $p_n(U) =: U'$ is again an open subset on the torus T_n. Suppose that $f : U \to Cl_{0n}(\mathbb{C})$ is an n-fold periodic function then the projection p_n induces a well-defined function $p_n(f) =: f' : U' \to Cl_n(\mathbb{C})$ on the n-torus defined by $f(p_n^{-1}(\mathbf{x})')$ for each $\mathbf{x}' \in U'$. In the cases where $\Im(\lambda) \neq 0$, for each $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n \backslash \Omega$ the associated functions $\wp_{\lambda;0,\ldots,0}(\mathbf{y} - \mathbf{x}), \wp^h_{\lambda;0,\ldots,0}(\mathbf{y} - \mathbf{x})$ induce functions $G_\lambda(\mathbf{y}' - \mathbf{x}')$ resp. $G^h_\lambda(\mathbf{y}' - \mathbf{x}')$ on the n-torus where $\mathbf{x}' := p_n(\mathbf{x})$ and $\mathbf{y}' := p_n(\mathbf{y})$. These functions are defined on $(T_n \times T_n)\backslash diag(T_n \times T_n)$. The projection map p_n induces a projection of the operator $\mathbf{D} - \lambda$ to a differential operator $\mathbf{D}'_\lambda$ acting on differentiable functions on T_n. Functions satisfying $\mathbf{D}'_\lambda$ are then called toroidal λ-holomorphic. In the same way the projection map p_n induces a projection of the Helmholtz operator $\Delta + \lambda^2$ to a second order operator Δ'_λ which will be called the toroidal Helmholtz operator. Its null-solutions will be called toroidal λ-harmonic. Again in the case $\lambda \neq 0$ with $\Im(\lambda) = 0$ one takes $|\mathbf{m}| > \frac{n+1}{2}$ partial derivatives.

More generally, as explained in [15], the decomposition of the lattice Ω into the direct sum of the sublattices $\Omega_l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$ and $\Omega_{n-l} := \mathbb{Z}e_{l+1} + \cdots + Ze_l$ gives following [23] rise to n conformally inequivalent different spinor bundles $E^{(l)}$ on $T_n \cong R^n/\Omega$ by making the identification $(\mathbf{x}, X) \iff (x + \underline{m} + \underline{n}, (-1)^{m_1+\cdots+m_l} X)$ with $\mathbf{x} \in \mathbb{R}^n, X \in Cl_{0n}(\mathbb{C})$. In the cases $\Im(\lambda) \neq 0$, the projection p_n of the associated modifications of the λ-holomorphic $\wp$ function

$$\wp_{\lambda;0,\cdots,0;k,l}(\mathbf{x}) := \sum_{\underline{m}\in\Omega_l}\sum_{\underline{n}\in\Omega_{n-l}}(-1)^{m_1+\cdots+m_l}e_\lambda(\mathbf{x}+\underline{m}+\underline{n})$$

then defines well defined λ-holomorphic spinor sections on the associated spinor bundles E^l of the n-torus T_n. The function $\wp_{\lambda;0,\cdots,0;k,l}(\mathbf{x})$ satisfies $\wp_{\lambda;0,\cdots,0;k,l}(\mathbf{x}+\omega) = (-1)^{m_1+\cdots+m_l}\wp_{\lambda;0,\cdots,0;k,l}(\mathbf{x})$. Its projection under p_n will be denoted by $G_{\lambda,k,l}$. Similarly, the projection of the modified λ-harmonic function $\wp^h_{\lambda;0,\ldots,0}(\mathbf{x})$ defined by

$$\wp^h_{\lambda;0,\ldots,0;k,l}(\mathbf{x}) := \sum_{\underline{m}\in\Omega_l}\sum_{\underline{n}\in\Omega_{n-l}}(-1)^{m_1+\cdots+m_l}h_\lambda(\mathbf{x})$$

defines well defined spinor sections $G^h_{\lambda,k,l}$ that are in the kernel of the associated spinorial Helmholtz operator on the associated spinor bundles E^l of the n-torus T_n. In the case where we take $l = n$ and where we make the trivial identification (x, X) with $(x+\omega, X)$ we deal with the trivial bundle on T_n. In all what follows we restrict to formulate the results for the trivial bundle. For the other bundles the formulas can be adapted correspondingly, by substituting the functions G_λ by $G_{\lambda,k,l}$ resp. G^h_λ by $G^h_{\lambda,k,l}$. In the limit case where $\lambda \to 0$ we obtain the Cauchy kernel functions for monogenic functions on conformally flat cylinders and tori introduced in [14, 15].

4.2 Integral representation formulas

By the Borel-Pompeiu formula and Cauchy's integral formula for the operator $(\mathbf{D}-\lambda)$ in the setting of the Euclidean space $\mathbb{R}^n$, [29, 31, 33], we obtain the following Cauchy integral formula for toroidal λ-holomorphic functions:

Theorem 5 *Let $\Im(\lambda) \neq 0$. Suppose V' is a sub domain of a domain U' lying on T_n and suppose that V' has a compact closure $cl(V')$. Assume further that $cl(V') \subset U'$ and that $p_n^{-1}(\partial V')$ is a Lipschitz surface. Let $f' : U' \to Cl_n(\mathbb{C})$ be a toroidal λ-holomorphic function. Then we have for each $\mathbf{y} \in V'$*

$$f'(\mathbf{y}') = \int_{\partial V'} G_{-\lambda}(\mathbf{x}'-\mathbf{y}')(D_{\mathbf{x}}p_n n(\mathbf{x}))f'(\mathbf{x}')dS(\mathbf{x}'), \tag{11}$$

where $D_{\mathbf{x}}p_n$ is the derivative of p_n at $\mathbf{x}$.

Remark: In the case where $\lambda \neq 0$ with $\Im(\lambda) = 0$ we still get a Cauchy integral formula for the partial derivatives of orders $|\mathbf{m}| > \frac{n+1}{2}$.

Corollary 1 *Let $\lambda \neq 0$ with $\Im(\lambda) = 0$. Let $\mathbf{m}$ be a multi-index with $|\mathbf{m}| > \frac{n+1}{2}$. Suppose that V' is a sub domain of a domain U' lying on T_n and suppose that V' has a compact closure $cl(V')$. Assume further that $cl(V') \subset U'$ and that $p_n^{-1}(\partial V')$ is a Lipschitz surface. Let $f' : U' \to Cl_{0n}(\mathbb{C})$ be a toroidal λ-holomorphic function. Then we have for each $\mathbf{y} \in V'$*

$$f_{\mathbf{m}}{}'(\mathbf{y}') = \int_{\partial V'} G_{-\lambda,\mathbf{m}}(\mathbf{x}'-\mathbf{y}')(D_{\mathbf{x}}p_n n(\mathbf{x}))f'(\mathbf{x}')dS(\mathbf{x}'), \tag{12}$$

where $D_{\mathbf{x}}p_n$ is the derivative of p_n at $\mathbf{x}$.

In all that follows we restrict to explicitly formulate the results for the cases $\Im(\lambda) \neq 0$.

Suppose now that Σ is a sufficiently smooth hypersurface lying in T_n and that U' is a domain whose boundary is Σ. Let u be an arbitrary $Cl_{0n}(\mathbb{C})$ valued function belonging to $L^p(\Sigma)$. Then the integral

$$\int_{\Sigma} G_{-\lambda}(\mathbf{x}' - \mathbf{y}')(D_{\mathbf{x}}p_n n(\mathbf{x}))u(\mathbf{x}')dS(\mathbf{x}')$$

defines a toroidal λ-holomorphic function $f'(\mathbf{y}')$ on U'. Notice that we only claim that u' belongs to L^p. It does not necessarily need to have partial derivatives.
The toroidal λ-holomorphic function f' lifts to an n-fold periodic λ-holomorphic function defined on the n-fold periodic open set $U = p_n^{-1}(U')$. In the case where U contains all points of T_n except of a finite number of closed subsets, the lifted function f can be represented in the form given in Theorem 3. In the case of not dealing with non-essential singularities, the series over the orders N_i can extend up to $+\infty$. The particular case dealing with non-essential isolated poles is formulated explicitly below.

The projection map p_n also gives the following version of the Borel-Pompeiu formula for toroidal λ-holomorphic functions.

Theorem 6 *Let $\Im(\lambda) \neq 0$. Suppose that V' is a domain in T_n with compact closure and strongly Lipschitz boundary. Suppose also that $\theta : cl(V') \to Cl_{0n}(\mathbb{C})$ is a continuous function and that $\theta|_{V'}$ belongs to $C^1(V')$. Then for each $\mathbf{y}' \in V'$*

$$\theta(\mathbf{y}') = \Big(\int_{\partial V'} G_{-\lambda}(\mathbf{x}' - \mathbf{y}')(D_{\mathbf{x}}p_n n(\mathbf{x}))\theta(\mathbf{x})dS(\mathbf{x}') - \int_{V'} G_{-\lambda}(\mathbf{x}' - \mathbf{y}')\mathbf{D}'_{+\lambda}\theta(\mathbf{x}')d\mu(\mathbf{x}') \Big),$$

where μ is the projection of volume Lebesgue measure on $\mathbb{R}^n$ onto T_n.

Remark: In the case where $\theta|_{V'}$ is at least a $C^{|\mathbf{m}|}$ function with $|\mathbf{m}| > \frac{n+1}{2}$, we can formulate a similar result for the $|\mathbf{m}|$-th partial derivative for the case $\lambda \neq 0$ with $\Im(\lambda) = 0$.

Let again U' be a sub domain of T_n with compact closure and $\theta : U' \to Cl_{0n}(\mathbb{C})$ be an L^p function with $1 < p < \infty$. Again, by adapting the results from [33] obtained for the Euclidean space we readily obtain that on the n-torus holds

$$\mathbf{D}'_{\lambda} \int_{U'} G_{\lambda}(\mathbf{y}' - \mathbf{x}')\theta(\mathbf{x}')d\mu(\mathbf{x}') = \theta(\mathbf{y}')$$

for each $\mathbf{y}' \in U'$.

Finally, using the n-fold periodic basic function $\wp^{h}_{\lambda;0,\dots,0}$ for the Helmholtz operator, we obtain a Green's formula for solutions to the homogeneous generalized Helmholtz equation on the n-torus.

Theorem 7 *Let $\Im(\lambda) \neq 0$. Suppose that $h : U' \to Cl_{0n}(\mathbb{C})$ is a solution to the toroidal Helmholtz operator Δ'_λ on the domain $U' \subset T_n$. Suppose also that V' is a relatively compact subdomain of U' and that $cl(V') \subset U'$. Then provided the boundary of V' is sufficiently smooth*

$$h(\mathbf{y}) = \int_{\partial V'} (G_{-\lambda}(\mathbf{x}'-\mathbf{y}')(D_{\mathbf{x}} p_k n(\mathbf{x}))h(\mathbf{x}) + G^h_{-\lambda}(\mathbf{y}'-\mathbf{x}')(D_{\mathbf{x}} p_n n(\mathbf{x}))\mathbf{D}'_{+\lambda} h(\mathbf{x}'))dS(\mathbf{x}') \tag{13}$$

for each $\mathbf{y}' \in V'$.

Remark: In the limit case $\lambda \to 0$ we re-obtain the integral formulas for toroidal monogenic functions given in [14].

Finally, Theorem 2 and Theorem 4 allow us to set up the following important representation theorem, which we consider as one of the main results of this paper:

Theorem 8 *Let $\Im(\lambda) \neq 0$. Let $a'_1, a'_2, \ldots, a'_p \in T_n$ be a finite set of points. Suppose that $f' : T_n \backslash \{a'_1, \ldots, a'_p\} \to Cl_{0n}(\mathbb{C})$ is a toroidal λ-holomorphic function (resp. a function in the kernel of the toroidal Helmholtz operator) which has at most isolated poles at the points a'_i of the order K_i. Then there exists Clifford numbers $b'_1, \ldots, b'_p \in Cl_{0n}(\mathbb{C})$ such that*

$$f'(\mathbf{x}') = \sum_{i=1}^{p} \sum_{m=0}^{K_i-(n-1)} \sum_{m=m_1+m_2+\cdots+m_n} \left[G_{\lambda,\mathbf{m}}(\mathbf{x}' - a'_i) b'_i \right] \tag{14}$$

in the λ-holomorphic case, or

$$f'(\mathbf{x}') = \sum_{i=1}^{p} \sum_{m=0}^{K_i-(n-1)} \sum_{m=m_1+m_2+\cdots+m_n} \left[G^h_{\lambda,\mathbf{m}}(\mathbf{x}' - a'_i) b'_i \right] \tag{15}$$

in the λ-harmonic case, respectively.

This result follows from applying the projection map p_n to the expressions of Theorem 2 and Theorem 4 which is an additive map. Similarly, one can adapt this theorem to the general case dealing with non-isolated singularities as mentioned in the previous section.

4.3 Plemelj-Sokhotzkij projection formulas

Throughout this section we suppose that Σ is an n-fold periodic Lipschitz hypersurface in $\mathbb{R}^n$ and that $\Im(\lambda) \neq 0$. Let Σ' denote the hypersurface $p_n(\Sigma) \subset T_n$. Suppose further that $u' \in L^p(\Sigma')$ for some p with $1 < p < \infty$. Then $u'(x')$, belongs to $L^p(\Sigma')$. As mentioned in the previous section, integral

$$[F^{T_n}_\lambda u'](\mathbf{y}') := \int_{\Sigma'} G_{-\lambda}(\mathbf{x}' - \mathbf{y}')(D_{\mathbf{x}} p_n n(\mathbf{x}))u'(\mathbf{x}')dS(\mathbf{x}'),$$

which we call the toroidal Cauchy transform, defines for each $\mathbf{x}' \in \Sigma'$ and each $\mathbf{y}' \in T_n \backslash \Sigma'$ a toroidal λ-holomorphic function on $T_n \backslash \Sigma'$. We now want to consider the case where $\mathbf{y}'$ is also an element of Σ'. In this case the Cauchy transform becomes a singular integral operator. We want to establish an analogy of the Plemelj-Sokhotzkij formulas. To this end we first show in analogy to the monogenic case treated in [14]:

Lemma 5 *The singular integral operator*

$$S_{\Sigma}^{T_n} : L^p(\Sigma) \to L^p(\Sigma) : [S_{\Sigma}^{T_n}(u)](\mathbf{y}) = P.V. \int_{\Sigma} \wp_{-\lambda,0,\ldots,0}(\mathbf{x}-\mathbf{y})n(\mathbf{x})u(\mathbf{x})dS(\mathbf{x}),$$

where $P.V.$ means the principal value of the integral in the sense of Cauchy, is a well defined L^p bounded operator for $1 < p < \infty$.

Proof. The integral

$$P.V. \int_{\Sigma} \sum_{\omega \in \Omega} e_{-\lambda}(\mathbf{x}-\mathbf{y}+\omega)n(\mathbf{x})u(\mathbf{x})dS(\mathbf{x})$$

is only singular for finitely many terms in the multi-periodic series

$$\sum_{\omega \in \Omega} e_{-\lambda}(\mathbf{x}-\mathbf{y}+\omega).$$

For these few terms the L^p boundedness follows from arguments describing the L^p boundedness for the double layer potential operator and singular Cauchy transform for Lipschitz graphs in $\mathbb{R}^n$ in [22] and elsewhere. For the remaining terms one simply notes that this part of the operator is a convolution with an L^∞ function. ■

Now we can establish

Theorem 9 *If $\mathbf{y}'(t)$ is a path in $T_n \backslash (\Sigma')$ with non-tangential limit $\mathbf{y}' \in \Sigma'$ then*

$$\lim_{t \to 0} \int_{\Sigma'} G_{-\lambda}(\mathbf{x}'-\mathbf{y}'(t))(D_{\mathbf{x}}p_n(n(\mathbf{x}))u'(\mathbf{x}')dS(\mathbf{x}')$$

$$= \pm \frac{1}{2}u'(\mathbf{y}') + P.V. \int_{\Sigma'} G_{-\lambda}(\mathbf{x}'-\mathbf{y}')(D_{\mathbf{x}}p_n n(\mathbf{x}))u'(\mathbf{x}')dS(\mathbf{x}')$$

for almost all $\mathbf{y}' \in \Sigma'$.

The plus or minus sign appearing in the previous formula depends on the choice of orientation one gives to Σ and on which side of Σ' the path $\mathbf{y}'(t)$ approaches $\mathbf{y}'$.

By adapting standard arguments from classical complex analysis in one variable we again can establish that the operators

$$\frac{1}{2}I \pm S_{\Sigma'}^{T_n} : L^p(\Sigma') \to L^p(\Sigma')$$

are well defined mutually annihilating idempotents, where

$$S_{\Sigma'}^{T_n}(u)(\mathbf{y}') = P.V. \int_{\Sigma'} G_{-\lambda}(\mathbf{x}'-\mathbf{y}')(D_{\mathbf{x}}p_n n(\mathbf{x}))u'(\mathbf{x}')dS(\mathbf{x}').$$

Remark: Let U' be a domain whose boundary is Σ'. Let $\mathbf{x}'$ be an element from the boundary surface Σ'. Then the previous theorem can be reformulated in the form

$$\lim_{\mathbf{y}' \to \mathbf{x}'} (F_\lambda^{T_n} u(\mathbf{y}')) = \frac{1}{2} u'(\mathbf{x}) + S_{\Sigma'}^{T_n}(u)(\mathbf{y}')$$

if $\mathbf{y}' \in U'$ and

$$\lim_{\mathbf{y}' \to \mathbf{x}'} (F_\lambda^{T_n} u(\mathbf{y}')) = \frac{1}{2} u'(\mathbf{x}) - S_{\Sigma'}^{T_n}(u)(\mathbf{y}')$$

if $\mathbf{y} \notin U'$.

5 Multi-periodic solutions to the Helmholtz equation on conformally flat cylinders

5.1 Definition and basic properties

Instead of considering the sum over a full n-dimensional lattice Ω we can also restrict the summation over a k-dimensional sublattice $\Omega_k = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k$ where $1 \leq k \leq n-1$. We introduce

Definition 3 *Let $k \in \{1, \ldots, n-1\}$ and Ω_k be the k-dimensional associated lattice as defined a above. In the cases $\Im(\lambda) \neq 0$, we define the λ-holomorphic generalized cotangent functions by the series*

$$\cot^{(k)}_{\lambda;0,\cdots,0}(\mathbf{x}) := \sum_{\omega \in \Omega_k} e_\lambda(\mathbf{x} + \omega). \tag{16}$$

Similarly, we define the λ-harmonic cotangent by

$$\cot^{(k)}_{\lambda;0,\cdots,0}(\mathbf{x})^h := \sum_{\omega \in \Omega_k} h_\lambda(\mathbf{x} + \omega) \tag{17}$$

satisfying the homogeneous Helmholtz equation $(\Delta + \lambda^2) \cot^{(k)}_{\lambda;0,\cdots,0}(\mathbf{x})^h = 0$ in each $\mathbf{x} \in \mathbb{R}^n \backslash \Omega_k$.

Remark: These series are of course normally convergent, since even the summation over the full n-dimensional lattice is normally convergent. In the limit case $\lambda \to 0$ we re-obtain the monogenic k-fold periodic cotangent functions introduced in [11]. Again, by applying Eisenstein's lemma, in the case where $\lambda \neq 0$ with $\Im(\lambda) = 0$ we get a convergent series when considering $|\mathbf{m}| > \frac{k+1}{2}$ partial derivatives of the fundamental solution in the summation. In what follows we again restrict to treat explicitly the case $\Im(\lambda) \neq 0$. In the case $\Im(\lambda) = 0$ one has again to work with partial derivatives of order $|\mathbf{m}| > \frac{k+1}{2}$.

Again, the projection map $p_k : \mathbb{R}^n \to \mathbb{R}^n/\Omega_k := C_k$ applied to the λ-holomorphic k-fold periodic function $\cot^{(k)}_{\lambda;0,\cdots,0}(\mathbf{y} - \mathbf{x})$ induces a well defined λ-holomorphic k-fold function on the manifolds C_k which are conformally flat cylinders.

For each $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n \backslash \Omega$ the functions $\cot^{(k)}_{\lambda;0,\ldots,0}(\mathbf{y}-\mathbf{x}), \cot^{(k)}_{\lambda;0,\ldots,0}(\mathbf{y}-\mathbf{x})^h$ induce functions $G^{(k)}_{\lambda}(\mathbf{y}'-\mathbf{x}')$ resp. ${G^{(k)}_{\lambda}}^h(\mathbf{y}'-\mathbf{x}')$ on the k-cylinder where again $\mathbf{x}' := p_k(\mathbf{x})$ and $\mathbf{y}' := p_k(\mathbf{y})$. The projection map p_k induces a projection of the operator $\mathbf{D}-\lambda$ to the operator $(\mathbf{D}-\lambda)'$ whose null-solution are called cylindrical λ-holomorphic. In the same way the projection map p_k induces a projection of the Helmholtz operator to the cylindrical Helmholtz operator.

Again, more generally, the decomposition of the lattice Ω_k into the direct sum of the sublattices $\Omega_l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$ and $\Omega_{k-l} := \mathbb{Z}e_{l+1} + \cdots + \mathbb{Z}e_l$ gives accordingly to [23] again rise to k conformally inequivalent different spinor bundles $E^{(l)}$ on $C_k \cong \mathbb{R}^n/\Omega_k$ by making the identification $(\mathbf{x}, X) \Longleftrightarrow (x+\underline{m}+\underline{n}, (-1)^{m_1+\ldots+m_l}X)$ with $\mathbf{x} \in \mathbb{R}^n, X \in Cl_{0n}(\mathbb{C})$. The projection p_k of the associated modifications of the projection of the λ-holomorphic (resp. λ-harmonic) cotangent function again define well defined λ-holomorphic spinor sections on the associated spinor bundles E^l of the k-cylinder C_k. Again, in the limit case $\lambda \to 0$ we re-obtain the monogenic spinor sections described in [15].

All the integral formulas including the projection formulas of the last section carry directly over to the context of the cylinders C_k, simply by replacing the functions G_λ and G^h_λ by $G^{(k)}_\lambda$ and ${G^h_\lambda}^{(k)}$. Finally we use these functions to set up explicit formulas for the cylindrical Helmholtz operator. Again in the case where $\Im(\lambda) = 0$ one has to work with partial derivatives of order $|\mathbf{m}| > \frac{k+1}{2}$ and of course restricted to the context of L^p functions which have a partial derivative of order $|\mathbf{m}| > \frac{k+1}{2}$.

5.2 The Dirichlet problem for the cylindrical Helmholtz operator

Suppose that $\Im(\lambda) \neq 0$ and that $1 \leq k \leq n-1$. Consider now the half-strip

$$S^+_k = [-\frac{1}{2}, \frac{1}{2}]e_1 \times \ldots \times [-\frac{1}{2}, \frac{1}{2}]e_k \times span\{e_{k+1}, \ldots, e_{n-1}\} \times \mathbb{R}^+ e_n.$$

We denote the half cylinder $p_k(S^+_k)$ by C^+_k. We introduce the notation $\mathbf{y}^{(n)}$ for the vector $y_1e_1 + \cdots + y_{n-1}e_{n-1} - y_ne_n$, where the minus sign in the last component is switched.
We claim that while in [32] and elsewhere one sees a development of basic ideas in classical harmonic analysis over the half space $\mathbb{R}^{n,+}$ a suitable analogue for this development in the context of cylinders is the half cylinder C^+_k. The Szegö kernel of λ-holomorphic functions $S^{(k)}_\lambda(\mathbf{x}', \mathbf{y}')$ for the half cylinder C^+_k is the function

$$S^{(k)}_\lambda(\mathbf{x}', \mathbf{y}') = G^{(k)}_\lambda(\mathbf{x}' - {\mathbf{y}^{(n)}}')D_{\mathbf{x}}p_k(e_n)$$

where $\mathbf{x}' \in \partial C^+_k$ and $\mathbf{y}' \in C^+_k$. The Poisson kernel for λ-harmonic functions for the half cylinder C^+_k then reads

$$P^{(k)}_\lambda(\mathbf{x}', \mathbf{y}') = 2\,\mathrm{Sc}(S^{(k)}_\lambda(\mathbf{x}', \mathbf{y}')).$$

In fact this follows directly from the Plemelj-Sokhotzkij projection formula. This Poisson kernel solves the Dirichlet problem for cylindrical homogeneous Helmholtz operator on C_k^+ with L^p data given on the boundary for $1 < p < \infty$.

In the particular case $k = n-1$ it holds $\partial C_{n-1}^+ = T_{n-1}$. The Poisson kernel thus solves the Dirichlet problem on the $n-1$ torus for the Helmholtz operator on the $n-1$-torus.

6 The inhomogeneous Helmholtz equation on cylinders and tori

Throughout this section suppose that V' is a sub domain of an open subset $U \subseteq T_n$ (resp. $U \subset C_k$ for $k = 1, \ldots, n$) and that the closure of V' has a strongly Lipschitz boundary $\partial V'$. For simplicity we also use the notation C_n for the n-torus T_n. Suppose that $f : V' \to Cl_{0n}(\mathbb{C})$ is a function belonging to the Sobolev space $W^p_{2,Cl_{0n}(\mathbb{C})}(V')$. In this section we exclusively suppose that $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$.
Again let $\Delta'_\lambda = p_k(\Delta + \lambda^2)$ be the associated cylindrical resp. toroidal Helmholtz operator. We introduce the cylindrical resp. toroidal Teodorescu transform by

$$T_\lambda^{C_k} : W^p_{l,Cl_{0n}(\mathbb{C})}(V') \to W^{p+1}_{l,Cl_{0n}(\mathbb{C})}(V');\ [T_\lambda^{C_k} f'(\mathbf{x})] = -\int\limits_{V'} G_{-\lambda}(\mathbf{x}' - \mathbf{y}') f'(\mathbf{x}') dV'(\mathbf{x}')$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are distinct points from V'. The cylindrical (toroidal) Cauchy transform has the mapping properties

$$F_\lambda^{C_k} : W^{p-1}_{l,Cl_{0n}(\mathbb{C})}(\partial V') \to W^p_{l,Cl_{0n}(\mathbb{C})}(V') \cap Ker\ \mathbf{D}'_\lambda;$$

$$[F_\lambda^{C_k} f'(\mathbf{y}')] = \int\limits_{\partial V'} G_{-\lambda}(\mathbf{x}' - \mathbf{y}') n(\mathbf{x}') D_{\mathbf{x}} p_k(e_n) f'(\mathbf{y}') dS'(\mathbf{x}').$$

Using the cylindrical (toroidal) Teodorescu transform, the Borel-Pompeiu formula (6) can now be reformulated in the classical form

$$f' = F_\lambda^{C_k} f' + T_\lambda^{C_k} \mathbf{D}'_\lambda f',$$

as formulated for the Euclidean case in [9, 10]. Adapting the arguments from [9] p. 80 that were explicitly worked out for the Euclidean case, one can show that the space of square integrable functions over a domain V' of the cylinder resp. of the n-torus, admits the orthogonal decomposition

$$L^2(V', Cl_{0n}(\mathbb{C})) = Ker\mathbf{D}'_\lambda \cap L^2(V', Cl_{0n}(\mathbb{C})) \oplus \mathbf{D}'_\lambda \overset{\circ}{W}{}^1_{2,Cl_{0n}(\mathbb{C})}(V').$$

The space $Ker\mathbf{D}'_\lambda \cap L^2(V', Cl_{0n}(\mathbb{C}))$ is a Banach space endowed with the L^2 inner product

$$\langle f', g' \rangle := \int_{V'} \overline{f(\mathbf{x}')}^\sharp g(\mathbf{x}') dV(\mathbf{x}'),$$

as used in [3].
As a consequence of the Cauchy integral formula that we established in Section 4.2 and Cauchy-Schwarz' equality we can show that this space has a continuous point evaluation and does hence possess a reproducing kernel $B(\mathbf{x}', \mathbf{y}')$, satisfying

$$f'(\mathbf{y}') = \int_{V'} B(\mathbf{x}', \mathbf{y}') f(\mathbf{x}') dV(\mathbf{x}') \quad \forall\, f' \in Ker \mathbf{D}'_\lambda \cap L_{2,Cl_{0n}(\mathbb{C})}(V').$$

Let f be an arbitrary function from $L^2(V', Cl_{0n}(\mathbb{C}))$. Then the operator

$$[P_\lambda^{C_k} f'(\mathbf{y}')] = \int_{V'} B(\mathbf{x}', \mathbf{x}') f(\mathbf{y}') dV(\mathbf{x}')$$

produces the ortho-projection from $L^2(V', Cl_{0n}(\mathbb{C}))$ into Ker $\mathbf{D}'_\lambda \cap L^2(V', Cl_{0n}(\mathbb{C}))$. It will be called the cylindrical (toroidal) λ-holomorphic Bergman projector. With these operators we can represent in complete analogy to the Euclidean case treated in [9] the solutions to the inhomogeneous Helmholtz equation on cylinders and tori. We establish

Theorem 10 *Let $\lambda \in \mathbb{C}$ with $\Im(\lambda) \neq 0$. Let V' be a domain on the flat cylinder C_k ($k = 1, \ldots, n-1$) resp. on the flat n-torus T_n. Let $f \in W^p_{2,Cl_{0n}(\mathbb{C})}(V')$ and $g \in W^{p+3/2}_{2,Cl_{0n}(\mathbb{C})}(\partial V')$. Let Δ'_λ stand for the cylindrical (toroidal) Helmholtz operator. Then the system*

$$\Delta'_\lambda u' = f' \quad in\ V' \tag{18}$$

$$u' = g' \quad at\ \partial V' \tag{19}$$

has a unique solution $u \in W^{p+2,loc}_{2,Cl_{0n}(\mathbb{C})}(V')$ of the form

$$u' = F_\lambda^{C_k} g' + T_{-\lambda}^{C_k} P_\lambda^{C_k} \mathbf{D}'_\lambda h' - T_{-\lambda}^{C_k}(I - P_\lambda^{C_k}) T_\lambda^{C_k} f' \tag{20}$$

where h' is the unique $W^{p+2}_{2,Cl_{0n}(\mathbb{C})}$ extension of g'.

To the proof one can apply the same calculation steps as in [9] pp. 81 involving now the properly adapted version of the Borel Pompeiu formula for cylindrical resp. toroidal λ-holomorphic functions and the adapted integral transform. Again, as in [9] p. 83 we can represent the cylindrical resp. toroidal Bergman projector in terms of algebraic expressions involving only the cylindrical (toroidal) Cauchy and Teodorescu transform, viz

$$P_\lambda^{C_k} = F_\lambda^{C_k}(tr T_\lambda^{C_k} F_\lambda^{C_k})^{-1} tr T_\lambda^{C_k},$$

where tr is the usual trace operator. This formula allows us to represent the solutions to the inhomogeneous cylindrical (toroidal) Helmholtz equation in terms of the singular integral operators that we introduced in the previous section.

Remark. It remains an open question how we can explicitly define the cylindrical and the toroidal λ-holomorphic Teodorescu and Cauchy transform for the cases where $\Im(\lambda) = 0$ with $\lambda \neq 0$. This would be necessary to develop an analogous representation formula as (20) for the solutions the inhomogeneous toroidal Helmholtz equation with real parameter $\lambda \neq 0$ on the torus. This represents a subject for further investigation.

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